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Recursive Dynamics Algorithm for Multibody Systems with Prescribed Motion

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Abstract

This paper uses spatial operator techniques to develop a new algorithm for the dynamics of multibody systems with hinges undergoing prescribed motion. This algorithm is spatially recursive and its computational complexity grows only linearly with the number of degrees of freedom in the system. Its structure is a hybrid of known recursive forward and inverse dynamics algorithms for regular multibody systems. Changes to the prescribed/non–prescribed nature of hinges can be implemented during run–time since they are handled with very low overhead in the algorithm.

Nomenclature

Coordinate–free spatial notation is used throughout this paper (see references [4,5] for additional details). The notation $\tilde{l}$ denotes the cross–product matrix associated with the 3–dimensional vector l. The notation x^* denotes the transpose of a matrix x. In the stacked notation used in this paper, indices are used to identify quantities pertinent to a specific body. Thus for instance, V denotes the vector of the spatial velocities for all the bodies in the serial–chain, and $V(k)$ denotes the spatial velocity vector of the k^{th} body. Some key quantities used in this paper are defined below.

n	number of bodies in the multibody system
$\mathcal{O}_k$	(inboard) body frame for the k^{th} body
$\mathcal{O}_k^+$	outboard frame on the $(k+1)^{th}$ body
$r(k)$	number of degrees of freedom for the k^{th} hinge

$\mathcal{N}$ $= \sum_{k=1}^{n} r(k)$, the overall degrees of freedom for the multibody system

$\theta(k)$ $\in \mathbb{R}^{r(k)}$, the vector of generalized coordinates for the k^{th} hinge

$\beta(k)$ $\in \mathbb{R}^{r(k)}$, the vector of generalized velocities for the k^{th} hinge

$l(k,j)$ $\in \mathbb{R}^3$, the vector from the k^{th} to the j^{th} body frame

$\phi(k,j)$ $\triangleq \begin{pmatrix} I & \tilde{l}(k,j) \\ 0 & I \end{pmatrix} \in \mathbb{R}^{6\times 6}$, the spatial transformation operator from the j^{th} hinge to the k^{th} body frame

$H^*(k)$ $\in \mathbb{R}^{6\times r(k)}$, the joint map matrix for the k^{th} hinge

$m(k)$ mass of the k^{th} body

$p(k)$ $\in \mathbb{R}^3$, the vector from $\mathcal{O}_k$ to the center of mass of the k^{th} body

$\mathcal{J}(k)$ $\in \mathbb{R}^{3\times 3}$, the inertia matrix for the k^{th} body about $\mathcal{O}_k$

$M(k)$ $= \begin{pmatrix} \mathcal{J}(k) & m(k)\tilde{p}(k) \\ -m(k)\tilde{p}(k) & m(k)I_3 \end{pmatrix} \in \mathbb{R}^{6\times 6}$, the spatial inertia of the k^{th} body referred to $\mathcal{O}_k$

$V(k)$ $= \begin{pmatrix} \omega(k) \\ v(k) \end{pmatrix} \in \mathbb{R}^6$, the spatial velocity of the k^{th} body referred to $\mathcal{O}_k$, with $\omega(k)$ and $v(k)$ denoting the angular and linear velocity components respectively

$a(k)$ $= \begin{pmatrix} \tilde{\omega}(k+1) & 0 \\ 0 & \tilde{\omega}(k+1) \end{pmatrix} [V(k) - V(k+1)] \in \mathbb{R}^6$, the Coriolis acceleration for the k^{th} body referred to $\mathcal{O}_k$

$b(k)$ $= \begin{pmatrix} \tilde{\omega}(k)\mathcal{J}(k)\omega(k) \\ m(k)\tilde{\omega}(k)\tilde{\omega}(k)p(k) \end{pmatrix} \in \mathbb{R}^6$, the gyroscopic spatial force for the k^{th} body referred to $\mathcal{O}_k$

$\alpha(k)$ $\in \mathbb{R}^6$, the spatial acceleration of the k^{th} body referred to $\mathcal{O}_k$

$f(k)$ $= \begin{pmatrix} N(k) \\ F(k) \end{pmatrix} \in \mathbb{R}^6$, the spatial force of interaction between the $(k+1)^{th}$ and the k^{th} body referred to $\mathcal{O}_k$, with $N(k)$ and $F(k)$ denoting the moment and force components respectively

$T(k)$ $\in \mathbb{R}^{r(k)}$, the generalized force for the k^{th} hinge

$\mathcal{M}$ $\in \mathbb{R}^{\mathcal{N}\times\mathcal{N}}$, the mass matrix for the multibody system

$\mathcal{C}$ $\in \mathbb{R}^{\mathcal{N}}$, the vector of Coriolis and centrifugal forces for the multibody system

$\mathcal{A}_r$, $\mathcal{A}_p$ the regular and prescribed multibody subsystems

$\mathcal{N}_r$, $\mathcal{N}_p$ the number of regular and prescribed degrees of freedom

$\mathcal{I}_r$, $\mathcal{I}_p$ the set of indices of the regular and prescribed hinges

Introduction

Dynamics simulations play an important role in the design and development of control systems for multibody systems such as spacecraft, robots and vehicles. Simulations are required for control algorithm validation as well as for hardware–in–the–loop testing. Due to the complexity of these dynamics simulations, a large amount of research over recent years has been devoted to the development of fast computational dynamics algorithms [6–10].

Attention has largely focused on the dynamics of *regular* multibody systems. For these systems, the dynamics problem requires solving the equations of motion for the set of generalized accelerations for a given state and set of applied generalized forces. The recently developed recursive algorithms are very efficient for solving this problem for systems with moderate to large number of degrees of freedom. They are commonly referred to as $O(\mathcal{N})$ algorithms because their computational complexity grows only linearly with the number of bodies in the system. For flexible multibody systems, the cost of the $O(\mathcal{N})$ flexible multibody dynamics algorithm has an additional quadratic (sometimes cubic) dependency on the number of deformation degrees of freedom. A number of these $O(\mathcal{N})$ algorithms have been developed independently by several researchers, and the relationship among them for rigid multibody systems has been studied in reference [5].

Multibody systems with prescribed motion differ from regular multibody systems in that some of the generalized accelerations are known *a priori*, while the corresponding generalized forces are unknown. Conventional approaches to handling prescribed motion treat the prescribed motion as additional global constraints on the system dynamics. These constraints are then used to either eliminate some of the degrees of freedom from the equations of motion, or additional Lagrange multipliers are used to account for these constraints.

In this paper, we develop a new $O(\mathcal{N})$ recursive algorithm for the dynamics of multibody systems which contain degrees of freedom undergoing prescribed motion. The structure of this algorithm is closely related to that of the conventional $O(\mathcal{N})$ dynamics algorithm and does not make use of global constraints. Its structure is a hybrid of known inverse and forward dynamics algorithms for regular multibody systems. When all the hinges are regular, the steps in the above algorithm reduce to the well known articulated inertias based $O(\mathcal{N})$ forward dynamics algorithm. On the other hand, when all the hinges are prescribed hinges, the steps in the algorithm reduce to the composite rigid body inertias based $O(\mathcal{N})$ inverse dynamics algorithm. In general, the steps in the algorithm consist of a combination of both inverse and forward dynamics computational steps.

We make extensive use of the *spatial operator algebra* [4] techniques for multibody dynamics. The notational compactness of the spatial operators reduces the complexity of the dynamics formulation and provides the tools necessary for developing the computational algorithm. For simplicity, we develop the algorithms for serial–chain rigid multibody systems, and later discuss extensions to general topology, flexible multibody systems.

Dynamics of a Regular Serial Multibody System

We consider a serial–chain rigid multibody system with n bodies. As shown in Figure 1, the bodies are numbered in increasing order from tip to base. The outer most body is denoted body 1, and the inner most body is denoted body n.

Each body has two frames denoted $\mathcal{O}_k$ and $\mathcal{O}_k^+$ attached to it. Frame $\mathcal{O}_k$ is on the inboard side and is designated the body frame for the k^{th} body. The k^{th} hinge connects the $(k+1)^{th}$ and k^{th} bodies and its motion defines the motion of frame $\mathcal{O}_k$ with respect to frame $\mathcal{O}_{k+1}^+$. Free space motion of the multibody system is handled by using a 6 degree of freedom hinge between the base body and the inertial frame. The k^{th} hinge is assumed to have $r(k)$ degrees of freedom where $1 \le r(k) \le 6$, and its vector of generalized coordinates is denoted $\theta(k)$. For simplicity, and without any loss in generality, we assume that the number of generalized velocities for the hinge is also $r(k)$, i.e., there are no local nonholonomic constraints on the hinge. The vector of generalized velocities for the k^{th} hinge is denoted $\beta(k) \in \mathbb{R}^{r(k)}$. The choice of the hinge angle rates $\dot{\theta}(k)$ for the generalized velocities $\beta(k)$ is often an obvious and convenient choice. However, when the number of hinge degrees of freedom is larger than one, alternative choices are often preferred since they simplify and decouple the kinematic and dynamic parts of the equations of motion. One such instance is the use of the relative angular velocity (rather than Euler angle rates) among the generalized velocities for the base–body of a free–flying system. The overall number of degrees of freedom for the system is given by $\mathcal{N} = \sum_{k=1}^{n} r(k)$.

The spatial velocity $V(k)$ of the k^{th} body frame $\mathcal{O}_k$ is defined as $V(k) = \begin{pmatrix} \omega(k) \\ v(k) \end{pmatrix} \in \mathbb{R}^6$, with $\omega(k)$ and $v(k)$ denoting the angular and linear velocities of $\mathcal{O}_k$. The spatial force of interaction $f(k)$ across the k^{th} hinge is denoted $f(k) = \begin{pmatrix} N(k) \\ F(k) \end{pmatrix} \in \mathbb{R}^6$, with $N(k)$ and $F(k)$ being the moment and force components respectively. The relative spatial velocity across the k^{th} hinge is given by $H^*(k)\beta(k)$ where $H^*(k) \in \mathbb{R}^{6 \times r(k)}$ is the joint map matrix for the hinge. The spatial inertia $M(k)$ of the k^{th} body referred to $\mathcal{O}_k$ is defined as

$$M(k) = \begin{pmatrix} \mathcal{J}(k) & m(k)\tilde{p}(k) \\ -m(k)\tilde{p}(k) & m(k)I_3 \end{pmatrix} \in \mathbb{R}^{6\times 6}$$

where $m(k)$ is the mass, $p(k) \in \mathbb{R}^3$ is the vector from $\mathcal{O}_k$ to the center of mass, and $\mathcal{J}(k) \in \mathbb{R}^{3\times 3}$ is the inertia of the k^{th} body about $\mathcal{O}_k$.

Spatial operators have been used in the past as analysis tools, and to obtain compact representations of the equations of motion and key dynamical quantities for multibody systems [4]. In addition, the operators have the advantage that high-level operator expressions can be directly mapped into fast recursive algorithms, and the explicit computation of the operators is rarely required. For these reasons we adopt the spatial operator approach in the remainder of this paper.

The vector $\theta \stackrel{\triangle}{=} [\theta^*(1), \; \cdots \; \theta^*(n)]^* \in \mathbb{R}^{\mathcal{N}}$ denotes the vector of generalized coordinates for the system. Similarly, we define the vectors of generalized velocities $\beta \in \mathbb{R}^{\mathcal{N}}$ and generalized (hinge) forces $T \in \mathbb{R}^{\mathcal{N}}$ for the system. The vector of spatial velocities V is defined as $V \stackrel{\triangle}{=} [V^*(1) \; \cdots \; V^*(n)]^* \in \mathbb{R}^{6n}$. The vector of spatial accelerations is denoted $\alpha \in \mathbb{R}^{6n}$, that of the Coriolis accelerations by $a \in \mathbb{R}^{6n}$, the body gyroscopic forces by $b \in \mathbb{R}^{6n}$, and the body interaction spatial forces by $f \in \mathbb{R}^{6n}$. Note that the components of the vectors a and b are nonlinear functions of the velocities and expressions for them are given in the nomenclature section. The equations of motion for the serial–chain multibody system can be written as follows (see reference [4] for details):

$$
\begin{aligned}
V &= \phi^* H^* \beta && (1)\\
\alpha &= \phi^*[H^* \dot{\beta} + a] && (2)\\
f &= \phi[M\alpha + b] && (3)\\
T &= Hf = \mathcal{M}\dot{\beta} + \mathcal{C} && (4)
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{M} &\triangleq H\phi M \phi^* H^* \in \mathbb{R}^{\mathcal{N}\times\mathcal{N}} && (5)\\
\mathcal{C} &\triangleq H\phi[M\phi^* a + b] \in \mathbb{R}^{\mathcal{N}} && (6)
\end{aligned}
$$

$H \triangleq \text{diag}\Big\{H(k)\Big\}$, $M \triangleq \text{diag}\Big\{M(k)\Big\}$,

$$
\mathcal{E}_\phi \triangleq \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ \phi(2,1) & 0 & \dots & 0 & 0 \\ 0 & \phi(3,2) & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \phi(n,n-1) & 0 \end{pmatrix} \quad (7)
$$

$$
\text{and} \quad \phi \triangleq (I - \mathcal{E}_\phi)^{-1} = \begin{pmatrix} I & 0 & \dots & 0 \\ \phi(2,1) & I & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \phi(n,1) & \phi(n,2) & \dots & I \end{pmatrix} \quad (8)
$$

with

$$
\phi(i,j) \triangleq \phi(i,i-1) \ \cdots \ \phi(j+1,j) \quad \text{for} \quad i > j
$$

The spatial transformation operator $\phi(k,j)$ is defined as $\begin{pmatrix} I & \tilde{l}(k,j) \\ 0 & I \end{pmatrix} \in \mathbb{R}^{6\times6}$ with $l(k,j) \in \mathbb{R}^3$, denoting the vector from the k^{th} to the j^{th} body frame. The notation $\tilde{l}$ denotes the cross–product matrix associated with the 3–dimensional vector l.

$\mathcal{M}$ is the *mass matrix* of the system, and the vector $\mathcal{C}$ contains the velocity dependent Coriolis and gyroscopic hinge forces. The operator expressions in (1)–(4) can be implemented in the form of a recursive computational algorithms for the inverse dynamics of the system. This algorithm is the same as the efficient Newton–Euler inverse dynamics algorithm developed in reference [11].

Additional external forces on the bodies in the system are handled by adding their effect to the component $b(.)$ vectors for the bodies.

We refer to a multibody system with no prescribed motion hinges as a *regular multibody system*. Regular multibody systems have provided the main focus for research for the development of efficient computational dynamics algorithms. For such systems, the vector of generalized forces T is provided as an input and the computation of the generalized accelerations vector $\dot{\beta}$ is desired. In other words, for a given vector T, the dynamics problem for regular systems requires the solution of (4) for the vector $\dot{\beta}$. A number of methods for solving this dynamics problem have been proposed (see reference [5] for a survey and comparison for rigid multibody systems). The $O(\mathcal{N})$ articulated body algorithm does not require the computation or inversion of the mass matrix and is highly efficient. Its complexity grows only linearly with the number of degrees of freedom in the system. In the remainder of this paper we develop an $O(\mathcal{N})$ dynamics algorithm for systems that are not regular, but rather possess degrees of freedom undergoing prescribed motion.

Multibody Systems with Prescribed Motion Hinges

We use the term *prescribed degree of freedom* to denote a degree of freedom undergoing prescribed motion, and the term *regular degree of freedom* for a degree of freedom not undergoing prescribed motion. In general, the component degrees of freedom of multiple degree of freedom hinges may be a mix of prescribed and regular degrees of freedom. However, we make the notationally convenient assumption that all the degrees of freedom in a hinge are either all prescribed or all regular degrees of freedom. This assumption imposes no loss in generality since any multiple degree of freedom hinge can be decomposed into an equivalent concatenation of individual single degree of freedom hinges. Thus we assume that a multibody system with hinges containing a mix of regular and prescribed degrees of freedom has been replaced by an equivalent multibody system model consisting only of hinges whose component degrees of freedom are either all prescribed or all regular. A hinge, all of whose component degrees of freedom are prescribed (regular), is referred to as a *prescribed (regular) hinge*.

The number of regular hinges in the system is denoted n_r, while $\mathcal{I}_r$ denotes the set of indices of the regular hinges in the system. $\mathcal{I}_p$ denotes the corresponding set of indices of the prescribed hinges and $n_p = (n - n_r)$ denotes the number of prescribed hinges in the system. The total number of regular degrees of freedom in the system is given by $\mathcal{N}_r$ $(= \sum_{k \in \mathcal{I}_r} r(k))$, while the total number of prescribed degrees of freedom in the system is given by $\mathcal{N}_p$ $(= \sum_{k \in \mathcal{I}_p} r(k))$. Note that $\mathcal{N}_p + \mathcal{N}_r = \mathcal{N}$.

We use the sets of hinge indices, $\mathcal{I}_p$ and $\mathcal{I}_r$, to decompose the multibody system into a pair of subsystems: the *prescribed system* $\mathcal{A}_p$, and the *regular system* $\mathcal{A}_r$. $\mathcal{A}_p$ is the $\mathcal{N}_p$ degree of freedom system resulting from freezing all the regular hinges (i.e. hinges whose index is in $\mathcal{I}_r$), while $\mathcal{A}_r$ is the $\mathcal{N}_r$ degree of freedom system resulting from freezing all the prescribed hinges i.e. hinges whose index is in the set $\mathcal{I}_p$). This decomposition is illustrated in Figure 2.

Let $\beta_p \in \mathbb{R}^{\mathcal{N}_p}$, $T_p \in \mathbb{R}^{\mathcal{N}_p}$ and $H_p^* \in \mathbb{R}^{6n \times \mathcal{N}_p}$ denote the vector of generalized velocities, the vector of hinge forces and the joint map matrix for system $\mathcal{A}_p$. Similarly, let $\beta_r \in \mathbb{R}^{\mathcal{N}_r}$, $T_r \in \mathbb{R}^{\mathcal{N}_r}$ and $H_r^* \in \mathbb{R}^{6n \times \mathcal{N}_r}$ denote the corresponding quantities for system $\mathcal{A}_r$. Note that the two vectors $\beta_p \in \mathbb{R}^{\mathcal{N}_p}$ and $\beta_r \in \mathbb{R}^{\mathcal{N}_r}$, represent a decomposition of the vector of generalized velocities β which is consistent with the sets $\mathcal{I}_p$ and $\mathcal{I}_r$ respectively. Similarly T_p and T_r are decompositions of T, and H_p^* and H_r^* are decompositions of H^*. Those columns of H^* that correspond to the prescribed hinges form the columns of H_p^*, while those that correspond to the regular hinges form the columns of H_r^*. Consequently, it follows that the relative spatial velocity across the hinges can be written as

$$H_p^* \beta_p + H_r^* \beta_r = H^* \beta \tag{9}$$

Equations of Motion

We use the pair of subsystems, $\mathcal{A}_p$ and $\mathcal{A}_r$, to rewrite the equations of motion in (1)–(4) in the following partitioned form:

$$\begin{pmatrix} \mathcal{M}_{pp} & \mathcal{M}_{pr} \\ \mathcal{M}_{pr}^* & \mathcal{M}_{rr} \end{pmatrix} \begin{pmatrix} \dot{\beta}_p \\ \dot{\beta}_r \end{pmatrix} + \begin{pmatrix} \mathcal{C}_p \\ \mathcal{C}_r \end{pmatrix} = \begin{pmatrix} T_p \\ T_r \end{pmatrix} \tag{10}$$

Here $i, j \in \{r, p\}$ and

$$\mathcal{M}_{ij} \stackrel{\triangle}{=} H_i \phi M \phi^* H_j^*, \text{ and } \mathcal{C}_i \stackrel{\triangle}{=} H_i \phi [b + M \phi^* a] \tag{11}$$

Note that in (10), the submatrices $\mathcal{M}_{pp}$ and $\mathcal{M}_{rr}$ are the mass matrices for the $\mathcal{A}_p$ and $\mathcal{A}_r$ subsystems respectively.

For the dynamics computation problem, the known vector quantities in (10) are the regular hinge forces T_r and the prescribed hinge accelerations $\dot{\beta}_p$. It is required to compute the unknown vectors of regular hinge accelerations $\dot{\beta}_r$ and the prescribed hinge forces T_p. A simple rearrangement of (10) converts it into the form

$$\begin{pmatrix} T_p \\ \dot{\beta}_r \end{pmatrix} = \begin{pmatrix} \mathcal{S}_{pp} & \mathcal{S}_{pr} \\ -\mathcal{S}_{pr}^* & \mathcal{S}_{rr} \end{pmatrix} \begin{pmatrix} \dot{\beta}_p \\ T_r \end{pmatrix} + \begin{pmatrix} \mathcal{C}_p - \mathcal{S}_{pr}\mathcal{C}_r \\ -\mathcal{S}_{rr}\mathcal{C}_r \end{pmatrix} \tag{12}$$

where

$$\begin{aligned} \mathcal{S}_{pp} &\stackrel{\triangle}{=} \mathcal{M}_{pp} - \mathcal{M}_{pr}\mathcal{M}_{rr}^{-1}\mathcal{M}_{pr}^* \\ \mathcal{S}_{pr} &\stackrel{\triangle}{=} \mathcal{M}_{pr}\mathcal{M}_{rr}^{-1} \\ \mathcal{S}_{rr} &\stackrel{\triangle}{=} \mathcal{M}_{rr}^{-1} \end{aligned} \tag{13}$$

In (12), the quantities to be computed appear on the left. The direct use of (12) to obtain $\dot{\beta}_r$ and T_p requires the computation of $\mathcal{M}$, the inversion of $\mathcal{M}_{rr}$ and the formation of various matrix/matrix and matrix/vector products. The complexity of this dynamics algorithm is a *cubic* polynomial in $\mathcal{N}_r$ and a *quadratic* polynomial in $\mathcal{N}_p$, and its cost can be large for moderate to large order multibody systems. Another approach to handling prescribed motion hinges is to append additional global constraints and associated Lagrange multipliers to the equations of motion. Apart from the increased computational cost, neither of these methods goes well with the localized computational structure of the efficient $O(\mathcal{N})$ recursive dynamics algorithms for regular systems. In the following sections, we develop a new $O(\mathcal{N})$ prescribed motion dynamics algorithm that overcomes these limitations, does not require the computation of $\mathcal{M}$, and whose cost is linear in both $\mathcal{N}_p$ and $\mathcal{N}_r$.

Spatial Operator Expression for $\mathcal{M}_{rr}^{-1}$

Since $\mathcal{M}_{rr}$ is the mass matrix of the regular subsystem $\mathcal{A}_r$, we use operator factorization and inversion techniques developed in reference [4] for regular multibody systems to obtain a spatial operator expression for $\mathcal{M}_{rr}^{-1}$. This is used to develop closed–form spatial operator expressions for $\mathcal{S}_{pp}$, $\mathcal{S}_{pr}$ and $\mathcal{S}_{rr}$.

First, we use the following recursive algorithm to define several quantities required to obtain the new operator factorization:

$$
\left\{
\begin{array}{l}
\qquad P^+(0) = 0 \\
\textbf{for } \boldsymbol{k} \;=\; \mathbf{1} \cdots \boldsymbol{n} \\
\qquad \begin{aligned} P(k) &= \phi(k,k-1)P^+(k-1)\phi^*(k,k-1) + M(k) \end{aligned} \\
\quad \left\{
\begin{array}{l}
\textbf{if } k \in \mathcal{I}_r \\
\qquad \begin{aligned}
D_r(k) &= H_r(k)P(k)H_r^*(k) \\
G_r(k) &= P(k)H_r^*(k)D_r^{-1}(k) \\
K_r(k+1,k) &= \phi(k+1,k)G_r(k) \\
\overline{\tau}_r(k) &= I - G_r(k)H_r(k)
\end{aligned} \\
\quad \textbf{else} \\
\qquad \overline{\tau}_r(k) = I_6 \\
\textbf{end if}
\end{array}
\right. \\
\qquad \begin{aligned}
P^+(k) &= \overline{\tau}_r(k)P(k) \\
\psi(k+1,k) &= \phi(k+1,k)\overline{\tau}_r(k)
\end{aligned} \\
\textbf{end loop}
\end{array}
\right.
\tag{14}
$$

I_6 above denotes the 6×6 identity matrix. The quantities defined in (14) are very similar to the articulated body quantities required for the $O(\mathcal{N})$ forward dynamics algorithm for regular serial–chain systems [5]. In fact, a subset of these quantities are precisely the articulated body quantities for the $\mathcal{A}_r$ subsystem.

The operator $P \in \mathbb{R}^{6n \times 6n}$ is defined as a block diagonal matrix with its k^{th} diagonal element being $P(k) \in \mathbb{R}^{6 \times 6}$. The quantities in (14) are also used to define the following spatial operators:

$$
\begin{aligned}
D_r &\triangleq H_r P H_r^* \in \mathbb{R}^{\mathcal{N}_r \times \mathcal{N}_r} \\
G_r &\triangleq P H_r^* D_r^{-1} \in \mathbb{R}^{6n \times \mathcal{N}_r} \\
K_r &\triangleq \mathcal{E}_\phi G_r \in \mathbb{R}^{6n \times \mathcal{N}_r} \\
\overline{\tau}_r &\triangleq I - G_r H_r \in \mathbb{R}^{6n \times 6n} \\
\mathcal{E}_\psi &\triangleq \mathcal{E}_\phi \overline{\tau}_r \in \mathbb{R}^{6n \times 6n}
\end{aligned}
\tag{15}
$$

The operators D_r, G_r and $\overline{\tau}_r$ are all block diagonal with diagonal elements given by $D_r(k), G_r(k)$ and $\overline{\tau}_r(k)$ respectively. The operators K_r and $\mathcal{E}_\psi$ are not block diagonal. However, their only nonzero block elements are the elements $K_r(k,k-1)$'s and $\psi(k,k-1)$'s respectively along the first subdiagonal. It is easy to verify from (14) that P satisfies the equation

$$
M = P - \mathcal{E}_\psi P \mathcal{E}_\psi^* = P - \mathcal{E}_\phi P \mathcal{E}_\psi^*
\tag{16}
$$

Now, define the lower–triangular operator $\psi \in \mathbb{R}^{6n \times 6n}$ as

$$\psi \stackrel{\triangle}{=} (I - \mathcal{E}_\psi)^{-1} \tag{17}$$

The block elements, $\psi(i,j) \in \mathbb{R}^{6\times 6}$, of ψ are given by:

$$\psi(i,j) \stackrel{\triangle}{=} \begin{cases} \psi(i,i-1) \ \cdots \ \psi(j+1,j) & \text{for} \quad i > j \\ I & \text{for} \quad i = j \\ 0 & \text{for} \quad i < j \end{cases}$$

The structure of the operators $\mathcal{E}_\psi$ and ψ is identical to that of the operators ϕ and $\mathcal{E}_\phi$, except that the elements are now $\psi(i,j)$ rather than $\phi(i,j)$.

The expression in (11) for the regular system mass matrix, $\mathcal{M}_{rr}$, is referred to as a Newton–Euler factorization. We now describe results for an alternative operator factorization, and inversion of $\mathcal{M}_{rr}$. Further details and proofs can be found in references [4, 5].

Lemma 0.1 *The innovations factorization of the mass matrix $\mathcal{M}_{rr}$ is given by*

$$\mathcal{M}_{rr} = [I + H_r\phi K_r] D_r [I + H_r\phi K_r]^* \tag{18}$$

∎

The factorization in Lemma 0.1 can be regarded as an LDL^* factorization of $\mathcal{M}_{rr}$. The closed form operator expression for the inverse of the factor $[I + H_r\phi K_r]$ is described in Lemma 0.2 below.

Lemma 0.2 $[I + H_r\phi K_r]^{-1} = [I - H_r\psi K_r]$ ∎

Combining Lemma 0.1 and Lemma 0.2 leads to the following closed form operator expression for the inverse of the mass matrix $\mathcal{M}_{rr}$.

Lemma 0.3 $\mathcal{M}_{rr}^{-1} = [I - H_r\psi K_r]^* D_r^{-1} [I - H_r\psi K_r]$

∎

The factorization in Lemma 0.3 can be regarded as an L^*DL factorization of $\mathcal{M}_{rr}^{-1}$.

Spatial Operator Expressions for $\dot{\beta}$ and T_p

The operator expression for the inverse of $\mathcal{M}_{rr}$ in Lemma 0.3, together with (13), leads to the following lemma describing new closed–form operator expressions for the $\mathcal{S}_{ij}$ matrices.

Lemma 0.4

$$\mathcal{S}_{rr} = [I - H_r\psi K_r]^* D_r^{-1}[I - H_r\psi K_r] \tag{19}$$

$$\mathcal{S}_{pr} = H_p\left\{\psi K_r + P\psi^* H_r^* D_r^{-1}[I - H_r\psi K_r]\right\} = H_p\left\{(\psi - P\Omega)K_r + P\psi^* H_r^* D_r^{-1}\right\} \tag{20}$$

$$\mathcal{S}_{pp} = H_p[\psi M\psi^* - P\Omega P]H_p^* = H_p\left[(\psi - P\Omega)P + P\tilde{\psi}^*\right]H_p^* \tag{21}$$

where

$$\Omega \stackrel{\triangle}{=} \psi^* H_r^* D_r^{-1} H_r \psi, \quad \tilde{\psi} = \psi - I \tag{22}$$

Proof: See the appendix. ∎

The expressions for the $\mathcal{S}_{ij}$ matrices in Lemma 0.4 require only the inverse of the block–diagonal matrix D_r – an inverse that is relatively easy to obtain. Using these expression sin (12) leads to closed–form operator expressions for the vectors of regular hinge accelerations $\dot{\beta}_r$, the prescribed hinge forces T_p, and the body accelerations α. These are described in the following lemmas.

Lemma 0.5 *The operator expression for the generalized accelerations vector $\dot{\beta}_r$ for the regular hinges is as follows:*

$$\dot{\beta}_r = [I - H_r\psi K_r]^* D_r^{-1}\left\{T_r - H_r\psi(K_r T_r + P[H_p^*\dot{\beta}_p + a] + b)\right\} - K_r^*\psi^*[H_p^*\dot{\beta}_p + a] \tag{23}$$

Proof: See the appendix. ∎

Lemma 0.6 *The operator expression for the generalized forces vector T_p for the prescribed hinges is as follows:*

$$T_p = H_p\left\{P\tilde{\psi}^*\left[H_p^*\dot{\beta}_p + a\right] + P\psi^* H_r^* D_r^{-1} T_r + \left(\psi - P\Omega\right)\left(K_r T_r + b + P[H_p^*\dot{\beta}_p + a]\right)\right\} \tag{24}$$

Proof: See the appendix. ∎

Lemma 0.7 *The operator expression for the vector of spatial accelerations α for all the bodies in the system is as follows:*

$$\alpha = \psi^* \left\{[H_p^*\dot{\beta}_p + a] + H_r^* D_r^{-1}\left[T_r - H_r\psi\left(K_r T_r + b + P[H_p^*\dot{\beta}_p + a]\right)\right]\right\} \tag{25}$$

Proof: See the appendix. ∎

The expressions for the generalized and body accelerations, $\dot{\beta}_r$ and α, closely resemble the corresponding expressions for regular multibody systems. The key difference is in the additional terms involving the prescribed motion quantity $H^*\dot{\beta}_p$.

Recursive $O(\mathcal{N})$ Computational Algorithm

We now use the operator expressions in Lemmas 0.5–0.7, to develop the recursive $O(\mathcal{N})$ algorithm for handling dynamics with prescribed motion. First, we define the intermediate quantities, z, ϵ_r, ν_r and α^+, and use them to simplify the expressions for $\dot{\beta}_r$, T_p and α.

Lemma 0.8 *We have*

$$\begin{aligned} \alpha &= \alpha^+ + H_p^*\dot{\beta}_p + a \\ \dot{\beta}_r &= \nu_r - K_r^*\alpha \\ T_p &= H_p f \end{aligned} \tag{26}$$

where

$$\begin{aligned} z &\stackrel{\triangle}{=} \psi\left[K_r T_r + b + P(H_p^*\dot{\beta}_p + a)\right] \\ \epsilon_r &\stackrel{\triangle}{=} T_r - H_r z \\ \nu_r &\stackrel{\triangle}{=} D_r^{-1}\epsilon_r \\ \alpha^+ &\stackrel{\triangle}{=} \psi^*\left[H_r^*\nu_r + \mathcal{E}_\psi^*(H_p^*\dot{\beta}_p + a)\right] \\ f &\stackrel{\triangle}{=} P\alpha^+ + z \end{aligned} \tag{27}$$

Proof: In the appendix. ∎

The ability to convert spatial operator expressions into fast recursive algorithms by inspection is one of the important advantages of the spatial operator approach. It is a direct consequence of the special structure of the operators such as ϕ and ψ. This is discussed in more detail in [4,5]. We use this feature to convert the closed–form operator expressions for the vectors $\dot{\beta}_r$ and T_p in Lemma 0.8 into a recursive $O(\mathcal{N})$ computational algorithm. This algorithm requires a recursive tip–to–base sweep followed by a base–to–tip sweep as described below:

$$\left\{\begin{array}{l} \quad z(0) = 0 \\ \textbf{for } \boldsymbol{k} \ = \ \mathbf{1} \cdots \boldsymbol{n} \\ \quad \left\{\begin{array}{rcl} \multicolumn{3}{l}{\textbf{if } k \in \mathcal{I}_p} \\ z(k) & = & \phi(k,k-1)z^+(k-1) + b(k) + P(k)[H^*(k)\dot{\beta}(k) + a(k)] \\ z^+(k) & = & z(k) \\ \multicolumn{3}{l}{\quad \textbf{else}} \\ z(k) & = & \phi(k,k-1)z^+(k-1) + b(k) + P(k)a(k) \\ \epsilon_r(k) & = & T(k) - H(k)z(k) \\ z^+(k) & = & z(k) + G_r(k)\epsilon_r(k) \\ \nu_r(k) & = & D_r^{-1}\epsilon_r(k) \\ \multicolumn{3}{l}{\textbf{end if}} \end{array}\right. \\ \textbf{end loop} \end{array}\right. \tag{28}$$

$$\left\{\begin{array}{l} \quad \alpha^+(n+1) = 0 \\ \textbf{for } \boldsymbol{k} \ = \ \boldsymbol{n} \cdots \mathbf{1} \\ \quad \alpha^+(k) \ = \ \phi^*(k+1,k)\alpha(k+1) \\ \quad \left\{\begin{array}{rcl} \multicolumn{3}{l}{\textbf{if } k \in \mathcal{I}_p} \\ f(k) & = & P(k)\alpha^+(k) + z(k) \\ T(k) & = & H(k)f(k) \\ \multicolumn{3}{l}{\quad \textbf{else}} \\ \dot{\beta}(k) & = & \nu(k) - G_r^*(k)\alpha^+(k) \\ \multicolumn{3}{l}{\textbf{end if}} \end{array}\right. \\ \quad \alpha(k) \ = \ \alpha^+(k) + H^*(k)\dot{\beta}(k) + a(k) \\ \textbf{end loop} \end{array}\right. \tag{29}$$

In summary, the above dynamics algorithm requires the following 3 steps:

1. The recursive computation of all the body velocities $V(k)$, the Coriolis terms $a(k)$, and the gyroscopic forces $b(k)$ using the recursive inverse dynamics computations described earlier.

2. The recursive computation of the articulated body quantities using the algorithm in (14).

3. The computation of the vectors $\dot{\beta}_r$ and T_p using the recursive algorithms in (28) and (29).

Note that the recursions in Step (2) can be combined and carried out in conjunction with the tip to base sweep in (28). The complexity of this algorithm is *linear* in both $\mathcal{N}_p$ and $\mathcal{N}_r$, i.e. it is *linear* in $\mathcal{N}$. The algorithm is therefore an $O(\mathcal{N})$ algorithm. The computational cost is directly proportional to the number of regular hinges, and decreases as additional hinges undergo prescribed motion. The overhead associated with transitions between regular and prescribed character of a hinge is very small for this algorithm. When such a transition occurs, the only change required is to update the prescribed/regular status of the hinge in the index sets $\mathcal{I}_r$ and $\mathcal{I}_p$. However, appropriate care must be taken while simulating these transitions to ensure that continuity is maintained and physical laws are not violated.

An interesting feature of this dynamics algorithm is that its structure is a hybrid of known inverse and forward dynamics algorithms for regular multibody systems. When all the hinges are regular, $\mathcal{I}_p$ is empty, and the steps in the above algorithm reduce to the well known $O(\mathcal{N})$ articulated body forward dynamics algorithm [4, 6]. In this case, $P(k)$ is the articulated body inertia of all the bodies outboard of the k^{th} body. On the other hand, when all the hinges are prescribed hinges, $\mathcal{I}_r$ is empty, and the steps in the algorithm reduce to the composite rigid body inertias based $O(\mathcal{N})$ inverse dynamics algorithm [5]. In this case, $P(k)$ is the composite rigid body inertia of all the bodies that are outboard of the k^{th} body. More generally, the dynamics algorithm consists of a combination of both inverse and forward dynamics computational steps.

Extensions to General Multibody Systems

Extensions needed for general systems such as those with tree and closed topology, as well as systems with flexible bodies, are relatively straightforward to describe because the spatial operator approach provides a unified formulation and recursive algorithms for such regular multibody systems [12, 13].

Reference [12] describes the $O(\mathcal{N})$ dynamics algorithm for regular rigid multibody systems with tree and closed topologies. The structure of the tree topology algorithm is identical to that for regular serial chain systems, except that during an inward sweep, quantities from all the incoming branches are summed up, while during an outward sweep, quantities are propagated independently along each outgoing branch. In order to handle prescribed motion hinges in such systems, the computational steps for the prescribed hinges need to be altered in the same manner as described above for prescribed motion hinges in serial–chain systems.

For regular closed topology systems, the $O(\mathcal{N})$ dynamics algorithm consists of a part which requires computing the dynamics of a system with tree topology, and a second part which handles the closure constraints. Only the first part needs to be changed to handle prescribed motion, and the changes are as described above for tree topology systems.

Reference [13] describes the $O(\mathcal{N})$ dynamics algorithm for regular flexible multibody systems. The structure of this algorithm is similar to that for rigid multibody systems. For each body, in addition to processing the hinge degrees of freedom, the algorithm also processes the deformation degrees of freedom for the body. This introduces an additional quadratic (sometimes cubic) dependency on the number of deformation degrees of freedom on the computational cost of the algorithm. Prescribed motion at the hinges for flexible multibody systems is handled by modifying only the steps that correspond to the hinge degrees of freedom in the manner described above for rigid multibody systems. The steps that correspond to the deformation degrees of freedom remain unaffected.

Application Example

The dynamics algorithm described in this paper has been implemented and verified in simulation and is being used as an integral part of the real–time, hardware–in–the–loop DARTS (Dynamics Algorithms for Real–Time Simulation) flexible multibody dynamics simulation software package for NASA's Cassini spacecraft [14].

The Cassini Spacecraft

Figure 3 shows a deployed Cassini spacecraft configuration. The Huygens Probe is at the rear of the spacecraft. At the top of the spacecraft are the high and low gain antennas. Three booms are attached to the upper equipment module. They carry the high precision scan platform, the magnetometer and the 10 meter plasma/radio wave antenna, and the turntable. The middle spacecraft structure contains the propulsion tanks carrying 68% of the spacecraft mass. At the bottom of the propulsion module is the lower equipment module which supports three radio-isotope thermoelectric generators for spacecraft power, four reaction wheels, the articulated Probe relay antenna, and the articulable main engine. The high precision scan platform articulates in two directions, one about the boom axis one about an orthogonal intermediate axis. The turntable rotates continuously about the boom axis at 0.1, 1.0 or 3.0 rpm. The Probe relay antenna has one degree of freedom about an axis parallel to the turntable boom. The main engines can be articulated about two axes along the high gain antenna beam direction.

There are a number of key attitude control functions the spacecraft must perform. Th spacecraft acquires sun and stars for inertial reference shortly after launch. Then it will stay Earth/Sun pointed for ground communication and thermal control. There are a number of propulsive maneuvers for plane changes and orbit insertion. During the science phase of the mission the Huygens Probe will be released into Titan, the data from the Probe will be collected and relayed by the spacecraft back to earth over a period of four years of intensive investigations of the Saturnian system. The primary attitude control sensors are the star sensors and the gyros located on the high precision scan platform. The key actuators are the electro-mechanical actuators for the high precision platform, the turntable, the main engines, and the reaction wheels; and the chemical propulsion thrusters for attitude control, and the main engines. During main engine firing, the gyros are used as the control sensor and they are separated from the main engine gimbal actuators by the spacecraft bus and boom which are nonrigid. Furthermore, the bus carries a large amount of liquid propellant. This sensor and actuator non-collocation problem is one that requires high fidelity dynamics simulation of the spacecraft for control design and testing.

The DARTS Dynamics Simulator

The DARTS spacecraft dynamics simulator makes use of the $O(\mathcal{N})$ recursive spatial algebra flexible multibody dynamics algorithm described in reference [13]. The low computational cost of this algorithm is well suited for real–time simulation. The dynamics model conists of a star–topology model with a central flexible body and several articulated appendages. The appendages include the platforms, the engines, the reaction wheels and pendulum models for the fuel slosh.

The DARTS algorithm includes the modifications described in this paper to handle hinges with prescribed motion. This feature is required to model velocity control for the engine thrust vector as well as for simulating actuator faults. A direct velocity engine control model is used due to the high rate and large control authority of the engine controller. Also, its use simplifies the modeling of the complex engine actuation mechanism. Fault–protection software is tested by simulating faults during hardware–in–the–loop real–time simulations. The prescribed motion algorithm makes it easy to simulate actuator lock–up failures without requiring any redesign of the algorithm or modification of the embedded software. Actuator lock–up failures are modeled by switching the actuator hinge from regular to prescribed motion mode during run–time, and prescribing zero motion for the hinge. Appropriate care is used during these transitions to ensure continuity and physical correctness of the simulation. The low overhead of the algorithm makes the implementation of the transitions quite straightforward. Possible use of the prescribed motion feature for modeling fuel slosh and the probe release sequence from the spacecraft is under investigation.

Conclusions

This paper describes a recursive computational algorithm for the dynamics of multibody systems with prescribed motion. The structure of the algorithm is a hybrid of well known inverse and forward dynamics algorithms for regular multibody systems. The algorithm differs significantly from the traditional approaches which require either the partitioning of the mass matrix, or the use of global constraints to handle the prescribed motion. It offers several advantages: (a) its complexity is linear in the number of degrees of freedom in the system; (b) its structure is very closely related to that of the $O(\mathcal{N})$ recursive algorithm for regular systems; (c) no computation of the mass matrix or the explicit use of constraints is required; (d) its overhead is small, so that hinge models can change from prescribed to regular motion during run–time; (e) the algorithm allows the component degrees of freedom of a multiple degree of freedom hinge to be a mix of regular and prescribed models.

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Proofs of the Lemmas

Lemma Appendix :.9 *The following spatial operator identities are used in the proofs of the lemmas in this paper:*

$$\begin{aligned} \psi^{-1} &= \phi^{-1} + K_r H_r && (30) \\ \psi K_r H_r \phi &= \phi - \psi && (31) \\ [I - H_r \psi K_r] H_r \phi &= H_r \psi && (32) \\ \phi K_r [I - H_r \psi K_r] &= \psi K_r && (33) \\ \phi M \psi^* &= \tilde{\phi} P + P \psi && (34) \\ (\psi - P\Omega) M \phi^* &= (\psi - P\Omega) P + P \tilde{\psi}^* && (35) \end{aligned}$$

Proof: From (17), we have that

$$\psi^{-1} = I - \mathcal{E}_\psi = (I - \mathcal{E}_\phi) + \mathcal{E}_\phi G_r H_r = \phi^{-1} + K_r H_r$$

This identity immediately leads to the identities (30) and (31). The identities in (32) and (33) follow easily from (31).

Pre– and post–multiplying (16) from the left and the right by the operators ϕ and ψ respectively leads to (34).

We have that

$$\begin{aligned} (\psi - P\Omega) M \phi^* &= (I - P\psi^* H_r^* D_r^{-1} H_r) \psi M \phi^* \\ &= (I - P\psi^* H_r^* D_r^{-1} H_r)[\psi P + P \tilde{\phi}^*] \quad \text{(using (34))} \\ &= (\psi - P\Omega) P + P[\tilde{\phi}^* - \psi^* H_r^* K_r^* \phi^*] \\ &= (\psi - P\Omega) P + P \tilde{\psi}^* \quad \text{(using (31))} \end{aligned}$$

This establishes (35). ∎

Proof of Lemma 0.4:

Firstly, (19) is merely a restatement of Lemma 0.3. With regards (20), we see from (13) that

$$\begin{aligned}
\mathcal{S}_{pr} &= \mathcal{M}_{pr}\mathcal{M}_{rr}^{-1} \\
&= H_p\phi M\phi^* H_p^*[I - H_r\psi K_r]^* D_r^{-1}[I - H_r\psi K_r] \\
&= H_p\phi M\psi^* H_p^* D_r^{-1}[I - H_r\psi K_r] \quad \text{(using (32))} \\
&= H_p\left\{\phi K_r + P\psi^* H_p^* D_r^{-1}\right\}[I - H_r\psi K_r] \quad \text{(using (34))} \\
&= H_p\left\{\psi K_r + P\psi^* H_p^* D_r^{-1}[I - H_r\psi K_r]\right\} \quad \text{(using (33))} \\
&= H_p\left\{(\psi - P\Omega)K_r + P\psi^* H_p^* D_r^{-1}\right\}
\end{aligned}$$

For (21) we have from (13) that

$$\begin{aligned}
\mathcal{S}_{pp} &= \mathcal{M}_{pp} - \mathcal{M}_{pr}\mathcal{M}_{rr}^{-1}\mathcal{M}_{pr}^* = \mathcal{M}_{pp} - \mathcal{S}_{pr}\mathcal{M}_{pr}^* \\
&= H_p\left\{\phi - \psi K_r H_r^*\phi - P\psi^* H^* D_r^{-1}[I - H_r\psi K_r]H_r\phi\right\} M\phi^* H_p^* \\
&= H_p\left\{\psi - P\psi^* H^* D_r^{-1} H_r\psi\right\} M\phi^* H_p^* \quad \text{(using (31) and (32))} \\
&= H_p\left\{\psi - P\Omega\right\} M\phi^* H_p^* \\
&= H_p\left\{(\psi - P\Omega)P + P\tilde{\psi}^*\right\} H_p^* \quad \text{(using (35))}
\end{aligned}$$

∎

Proof of Lemma 0.5:

From (12) we have that

$$\begin{aligned}
\dot{\beta}_r &= \mathcal{S}_{rr}[T_r - \mathcal{C}_r] - \mathcal{S}_{pr}^*\dot{\beta}_p \\
&= [I - H_r\psi K_r]^* D_r^{-1}\left\{T_r - H_r\psi\left(K_r T_r + Pa + b\right)\right\} - K_r^*\psi^* a - \\
&\qquad \left\{K_r^*\psi^* + [I - H_r\psi K_r]^* D_r^{-1} H_p\psi P\right\} H_p^*\dot{\beta}_p \\
&= [I - H_r\psi K_r]^* D_r^{-1}\left\{T_r - H_r\psi\left(K_r T_r + P[H_p^*\dot{\beta}_p + a] + b\right)\right\} - K_r^*\psi^*[H_p^*\dot{\beta}_p + a]
\end{aligned}$$

∎

Proof of Lemma 0.6:

It follows from (12) that

$$
\begin{aligned}
T_p &= \mathcal{S}_{pp}\dot{\beta}_p + \mathcal{S}_{pr}[T_r - \mathcal{C}_r] + \mathcal{C}_p \\
&= H_p\left\{(\psi - P\Omega)P + P\tilde{\psi}^*\right\} H_p^*\dot{\beta}_p + H_p\left\{(\psi - P\Omega)K_r + P\psi^* H_p^* D_r^{-1}\right\} T_r - \\
&\qquad H_p\left\{\psi K_r + P\psi^* H_p^* D_r^{-1}[I - H_r\psi K_r]\right\} H_r\phi(M\phi^* a + b) + H_p\phi(M\phi^* a + b) \\
&= H_p\left\{(\psi - P\Omega)\left(K_r T_r + b + M\phi^* a + P H_p^*\dot{\beta}_p\right) + P\psi^* H_p^* D_r^{-1} T_r + P\tilde{\psi}^* H_p\dot{\beta}_p\right\} \\
&\qquad \text{(using (31) and (32))} \\
&= H_p\left\{P\tilde{\psi}^*\left[H_p^*\dot{\beta}_p + a\right] + P\psi^* H_r^* D_r^{-1} T_r + (\psi - P\Omega)\left(K_r T_r + b + P[H_p^*\dot{\beta}_p + a]\right)\right\} \\
&\qquad \text{(using (35))}
\end{aligned}
$$

∎

Proof of Lemma 0.7:

From (2) it follows that the vector of spatial accelerations α for the bodies is given by the expression

$$
\begin{aligned}
\alpha &= \phi^*[H^*\dot{\beta} + a] = \phi^*[H_p^*\dot{\beta}_p + H_r^*\dot{\beta}_r + a] \\
&= \phi^* H_r^*\left[[I - H_r\psi K_r]^* D_r^{-1}\left\{T_r - H_r\psi\left(K_r T_r + P[H_p^*\dot{\beta}_p + a] + b\right)\right\} - K_r^*\psi^*[H_p^*\dot{\beta}_p + a]\right] + \\
&\qquad \phi^*[H_p^*\dot{\beta}_p + a] \\
&= \psi^* H_r^* D_r^{-1}\left\{T_r - H_r\psi\left(K_r T_r + P[H_p^*\dot{\beta}_p + a] + b\right)\right\} - (\phi^* - \psi^*)[H_p^*\dot{\beta}_p + a] + \phi^*[H_p^*\dot{\beta}_p + a] \\
&\qquad \text{(using (31) and (32))} \\
&= \psi^*\left\{[H_p^*\dot{\beta}_p + a] + H_r^* D_r^{-1}\left[T_r - H_r\psi\left(K_r T_r + b + P[H_p^*\dot{\beta}_p + a]\right)\right]\right\}
\end{aligned}
$$

∎

Proof of Lemma 0.8:

From (23), and the definitions of z, ϵ, ν_r and α^+ it follows that

$$
\alpha = \psi^*[H_r^*\nu_r + H_p^*\dot{\beta}_p + a] = \alpha^+ + H_p^*\dot{\beta}_p + a
$$

Also,

$$
\dot{\beta}_r = [I - H_r\psi K_r]^* D_r^{-1}\nu_r - K_r^*\psi^*[H_p^*\dot{\beta}_p + a] = \nu_r - K_r^*\alpha
$$

In the case of T_p, we have that

$$
\begin{aligned}
T_p &= H_p\left\{P\tilde{\psi}^*\left[H_p^*\dot{\beta}_p + a\right] + P\psi^* H_r^* D_r^{-1} T_r + z - P\psi^* H_r^* D_r^{-1} H_r z\right\} \\
&= H_p\left\{P\tilde{\psi}^*\left[H_p^*\dot{\beta}_p + a\right] + P\psi^* H_r^*\nu_r + z\right\} = H_p\left\{P\alpha^+ + z\right\}
\end{aligned}
$$

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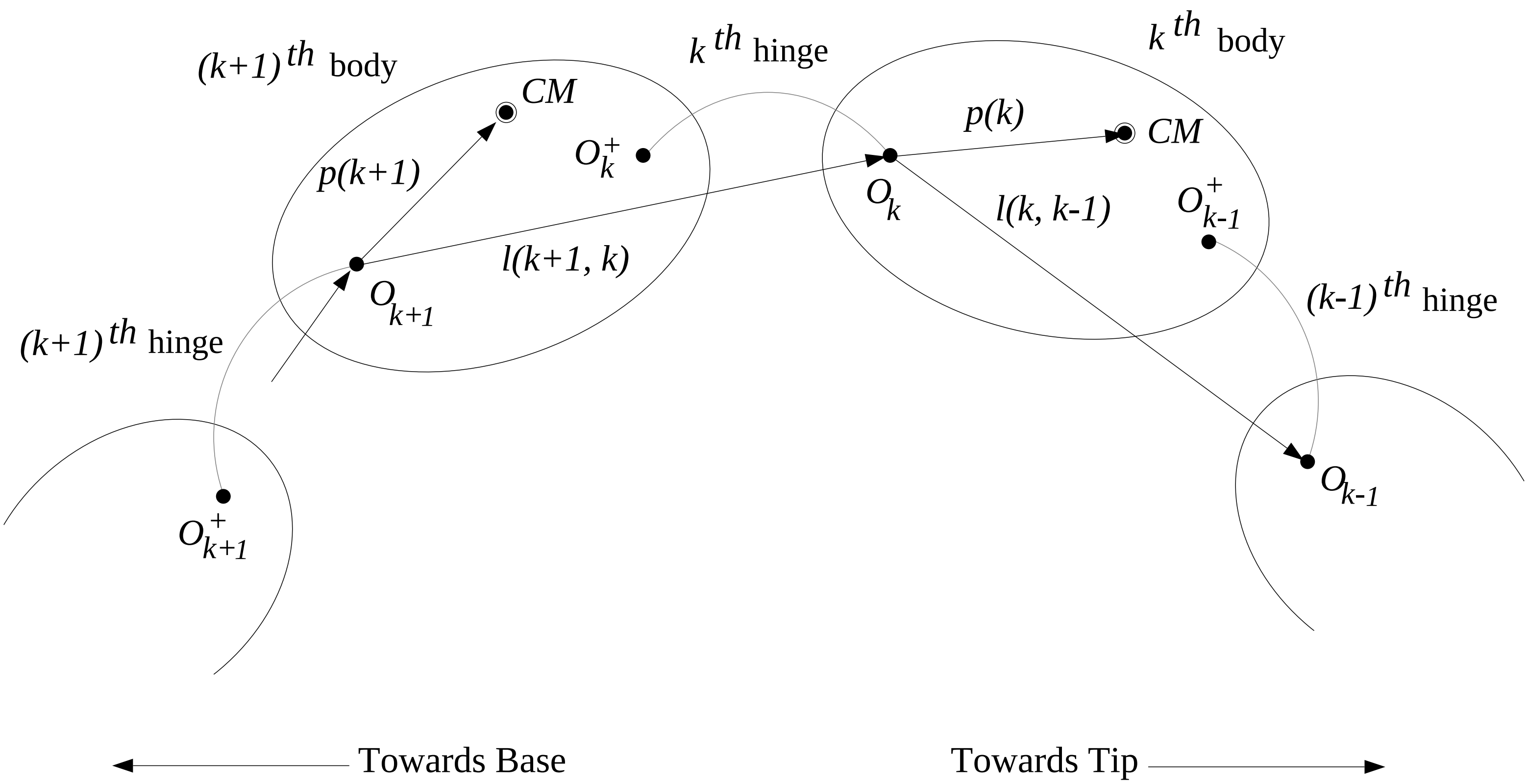

Figure 1: Illustration of the bodies and hinges in a serial–chain multibody system

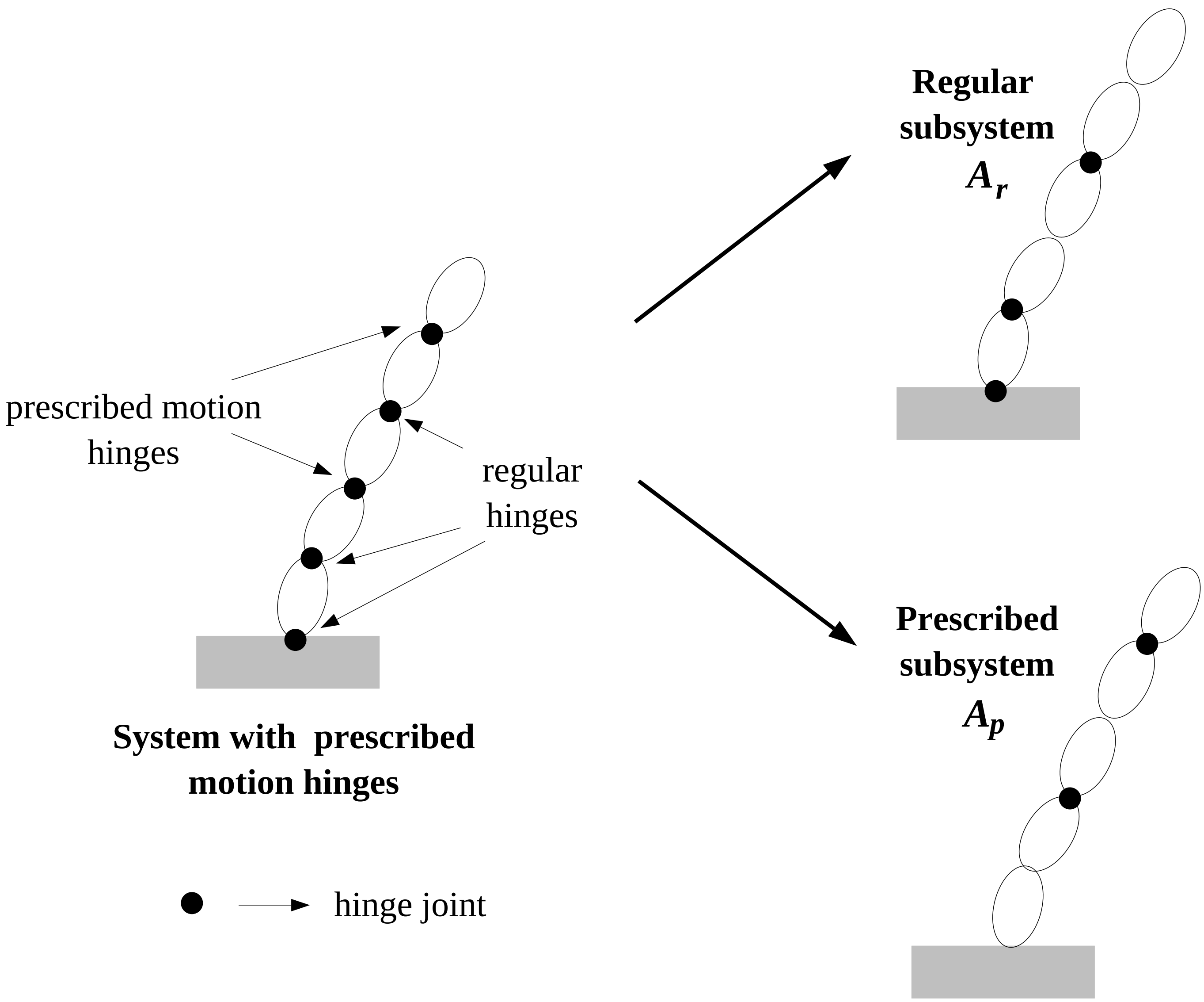

Figure 2: Illustration of the decomposition of the serial–chain multibody system

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Figure 3: A schematic of the Cassini spacecraft